Improved quadratic Gowers uniformity for the von Mangoldt function

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Green-Tao Theorem

Theorem

For each positive integer $k > 0$, the primes contain a progression of the form $(x, x + y, x + 2y, \dots, x + (k-1)y)$.

How many k-term arithmetic progressions in primes are there up to $[N]$?

Counting kAPs in Primes

We should study

$$\sum_{n,d\le N} 1_P(n)1_P(n+d)1_P(n+2d)\cdots 1_P(n+(k-1)d).$$

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In view of $\sum_{n\in[N]} \Lambda(n) = n + o(n)$ (the prime number theorem), it turns out to be more convenient to count

$$\sum_{n,d \leq N} \Lambda(n)\Lambda(n+d)\Lambda(n+2d)\cdots\Lambda(n+(k-1)d)$$

where

$$\Lambda(n) = \begin{cases} \log(p) & n = p^k \\ 0 & \text{otherwise} \end{cases}.$$

Main term-error term

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$$[\text{Count of kAPs in primes}] = [\text{Main term}] + [\text{Error term}].$$

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- If Λ behaves like a uniform distribution,

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- But prime numbers are not "roughly uniformly distributed."

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- It's far more likely for primes to be $1 \pmod 3$ or $2 \pmod 3$ than $0 \pmod 3$.
- Main term should be "relatively simple" and should take into account these local obstructions.
- There are other things to watch out for.

Example

- Suppose $p = 3$. The projection of the distribution $\pmod 3$ that $(x, x+y, x+2y)$ are prime should not be expected to be the same as when $(x, x+y)$ are prime.

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- Suppose $p = 3$. The projection of the distribution $\pmod 3$ that $(x, x+y, x+2y)$ are prime should not be expected to be the same as when $(x, x+y)$ are prime.
- If $x \equiv 1 \pmod 3$ and $x + y \equiv 2 \pmod 3$, then $x + 2y \equiv 0 \pmod 3$.
- Otherwise, $(x, x+y, x+2y)$ should equidistribute across moduli $(a, b, 2b-a)$ where all $a, b, 2b-a$ are nonzero moduli, i.e. $(1,1,1)$, $(2,2,2)$.
- The distribution of moduli $\pmod 3$ of $(x, x+y)$ are $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$.

Granville's Model

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Rough numbers (numbers without small prime factors) also "equidistribute" across nonzero a (mod p), and can also detect local obstructions across correlations. Define

$$P(Q) = \prod_{p \le Q} p$$

$$\Lambda_Q(n) = \frac{P(Q)}{\phi(P(Q))} 1_{\gcd(n, P(Q)) = 1}$$

where $\phi(n)$ is the number of positive integers less than n that are relatively prime to n, $Q(N)$ a sufficiently slow growing function in N.

Granville's Model

Note: we can factor

$$\Lambda_Q(n) = \prod_{p \le Q} \frac{p}{p-1} 1_{\gcd(n,p)=1} := \prod_{p \le Q} \Lambda_p(n).$$

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By the Chinese Remainder Theorem, we get

$$\sum_{n,d \le N} \Lambda_Q(n)\Lambda_Q(n+d)\cdots\Lambda_Q(n+(k-1)d) =$$

$$N^2 \prod_{p \le Q} \frac{1}{N^2} \sum_{n,d \le N} \Lambda_p(n)\cdots\Lambda_p(n+(k-1)d) + [\text{Error Term}].$$

Main term-error term

Let

$$\beta_p = \mathbb{E}_{n \in \mathbb{Z}/p\mathbb{Z}} \Lambda_p(n) \cdots \Lambda_p(n + (k-1)d).$$

$$\approx \frac{1}{N^2} \sum_{n \in [N]} \Lambda_p(n) \cdots \Lambda_p(n + (k-1)d).$$

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Thus, expect main term to be $\mathfrak{S}_k N^2$ where

$$N^2 \prod_{p \le Q} \beta_p \approx N^2 \prod_p \beta_p := \mathfrak{S}_k N^2$$

and error terms to be small, i.e., we should expect

$$\sum_{n,d \le N} \Lambda(n)\Lambda(n+d) \cdots \Lambda(n+(k-1)d)$$

$$- \sum_{n,d \le N} \Lambda_Q(n) \cdots \Lambda_Q(n+(k-1)d) = o(N^2).$$

Results

For $k = 3$, this was proven by van der Corput using Fourier analysis in 1939.

Theorem (Green-Tao, Green-Tao, Green-Tao-Ziegler ∼ 2010)

$$\sum_{n,d \le N} \Lambda(n)\Lambda(n+d)\cdots\Lambda(n+(k-1)d) = \mathfrak{S}_k N^2 + o(N^2)$$

with

$$\beta_p = \begin{cases} \frac{p^{k-2}(p+1-k)}{(p-1)^{k-1}} & p > k \\ \frac{p^{k-2}}{(p-1)^{k-1}} & p \le k \end{cases}.$$

More general result

Can obtain a similar asymptotic for counts of linear forms $\phi_1(n), \cdots \phi_k(n)$ where ϕ_i don't differ by a constant:

$$\sum_{\vec{n} \in \mathbf{K}} \Lambda(\phi_1(\vec{n})) \cdots \Lambda(\phi_k(\vec{n})) =$$

$$\prod_{p \leq Q} \sum_{\vec{n} \in \mathbf{K}} \Lambda_p(\phi_1(\vec{n})) \cdots \Lambda_p(\phi_k(\vec{n})) + o(N^d)$$

$$= \beta_\infty \prod_p \beta_p + o(N^d)$$

where $\mathbf{K} \subseteq [N]^d = \{1, \dots, N\}^d$ is convex and β_∞ is the volume of $\mathbf{K}$.

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Theorem (Tao-Teräväinen, 2021)

$$\sum_{n,d} \Lambda(n) \cdots \Lambda(n+(k-1)d) = \mathfrak{S}_k N^2 + O\left(\frac{N^2}{\log\log(N)^c}\right)$$

Quantitative bounds

Theorem (L. 2023)

For any $A > 0$, we have

$$\sum_{n,d} \Lambda(n) \cdots \Lambda(n+3d) = \mathfrak{S}_k N^2 + O_A\left(\frac{N^2}{\log^A(N)}\right)$$

constant in front of $\log^{-A}(N)$ is ineffective for the same reason as van der Corput's result.

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- Forms $\phi_1, \ldots, \phi_k$ that are not linearly independent, $\phi_1^{\otimes 2}, \cdots, \phi_k^{\otimes 2}$ also not linearly independent, but $\phi_1^{\otimes 3}, \cdots, \phi_k^{\otimes 3}$ are linearly independent (true complexity 2).

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- Forms $\phi_1, \ldots, \phi_k$ that are not linearly independent, $\phi_1^{\otimes 2}, \cdots, \phi_k^{\otimes 2}$ also not linearly independent, but $\phi_1^{\otimes 3}, \cdots, \phi_k^{\otimes 3}$ are linearly independent (true complexity 2).
- Follows from (very difficult) work of Manners (2021).

APs with shifted primes

Via the **W-trick**, we can show that

Theorem (Tao-Teräväinen 2021)

Suppose a subset $A \subseteq [N]$ doesn't contain any k-term arithmetic progressions of the form $(x, x+p-1, \ldots, x+(k-1)(p-1))$ where p is any prime. Then $|A| \ll N \log\log\log\log^{-c}(N)$.

For $k = 2$, can take bounds of N^{1-c} (Green 2022). For $k = 3$ can take $N \exp(-O(\log\log\log(N)^c))$ and for $k = 4$ can take $N \log\log\log^{-c}(N)$.

3APs with shifted primes

By assuming non-existence of Siegel zeros, we can improve the bounds for $k = 3$:

Theorem (L. 2023)

Assume (Landau)-Siegel zeros don't exist. Suppose a subset $A \subseteq [N]$ doesn't contain any 3-term arithmetic progressions of the form $(x, x + p - 1, x + 2(p - 1))$ where p is any prime. Then $|A| \ll N \exp(-O(\log \log^c(N)))$.

Though it may be possible to unconditionally show that

$$|A| \ll N \log^{-c}(N).$$

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- (a modern rendition of) van der Corput's (or rather Vinogradov's) proof is based on **Fourier analysis** and uses *Vaughan-type bilinear decompositions* of Λ to produce cancellation in phase.
- Fourier analysis can see **linear relations** such as $(x + 2y) = 2(x + y) - x$.
- It can't detect **quadratic relations** such as $(x + 3y)^2 - 3(x + 2y)^2 + 3(x + y)^2 - x^2 = 0$.

Gowers norms

$$\|f\|_{U^1(\mathbb{Z})}^2 := \left|\sum_{n,h\in\mathbb{Z}} f(n)\overline{f(n+h)}\right| = \left|\sum_n f(n)\right|^2$$

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$$\|f\|_{U^2(\mathbb{Z})}^4 := \left|\sum_{n,h_1,h_2\in\mathbb{Z}} f(n)\overline{f(n+h_1)f(n+h_2)}f(n+h_1+h_2)\right|$$

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We can rewrite as

$$\left|\sum_{n,h_1,h_2} \Delta_{h_1,h_2} f(n)\right|$$

where $\Delta_h f(n) = \overline{f(n+h)}f(n)$,
$\Delta_{h_1,h_2} f(n) = \Delta_{h_1}(\Delta_{h_2} f(n))$.

Gowers norms

So we define

$$\|f\|_{U^{s+1}(\mathbb{Z})}^{2^{s+1}} := \left| \sum_{n,h_1,\dots,h_{s+1}} \Delta_{h_1,\dots,h_{s+1}} f(n) \right|$$

and we define

$$\|f\|_{U^{s+1}([N])} = \frac{\|f 1_{[N]}\|_{U^{s+1}(\mathbb{Z})}}{\|1_{[N]}\|_{U^{s+1}(\mathbb{Z})}}.$$

We can verify that these are norms (except U^1)

Generalized von Neumann Theorem

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Generalized von Neumann Theorem

It can be shown that $\|f\|_{U^2([N])} \approx N^{-3/4}\|\hat{f}\|_{L^4(\mathbb{T})}$. This complements

Theorem (Gowers 2001)

For one-bounded $f_1, \dots, f_k$

$$\left|\frac{1}{N^2}\sum_{n,d} f_1(n)f_2(n+d)\cdots f_k(n+(k-1)d)\right| \ll$$

$$\min_i \|f_i\|_{U^{k-1}([N])}.$$

since obstructions to $U^2([N])$ being small are Fourier phases and hence explains van der Corput's approach.

Generalized von Neumann Theorem

Writing $\Lambda = (\Lambda - \Lambda_Q) + \Lambda_Q$, we obtain

$$\sum_{n,d} \Lambda(n)\Lambda(n+d)\cdots\Lambda(n+(k-1)d)$$

$$-\sum_{n,d} \Lambda_Q(n)\Lambda_Q(n+d)\cdots\Lambda_Q(n+(k-1)d)$$

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is $2^k - 1$ terms; each term has one term equal to $\Lambda - \Lambda_Q$. Thus, we want to prove that

$$\|\Lambda - \Lambda_Q\|_{U^{s+1}([N])}$$

is small.

Inverse Theorem

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Theorem (Green-Tao-Ziegler)

Suppose $\|f\|_{U^{s+1}([N])} \geq \delta$. Then there exists a degree $\leq s$ **nilsequence** *$F(g(n)\Gamma)$ with* **parameter** *$P(\delta)$,* **dimension** *$D(\delta)$ and* **complexity** *$M(\delta)$ such that*

$$|\mathbb{E}_{n \in [N]} f(n) F(g(n)\Gamma)| \gg_{\delta,s} 1.$$

A **nilmanifold** G/Γ is a topological quotient of a nilpotent Lie group G by a discrete cocompact subgroup Γ. A **polynomial sequence** $g(n)$ is a certain degree $\leq s$ "nice sequence" and $F : G/\Gamma \to \mathbb{C}$ a Lipschitz function.

Example

$G = \mathbb{R}^d$, $\Gamma = \mathbb{Z}^d$, let $P_1, \dots, P_d \in \mathbb{R}[x]$. Let $F : G/\Gamma \to \mathbb{C}$ be a smooth function. Let $g(n) = (P_1(n), \dots, P_d(n))$. An example of a nilsequence is

$$F(g(n)\Gamma) = F(P_1(n), \dots, P_d(n)).$$

For example, $e^{2\pi i \sqrt{2} n^2}$ is a nilsequence.

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$$g(n) = \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & \alpha n & \gamma n + \binom{n}{2}\alpha\beta \\ 0 & 1 & \beta n \\ 0 & 0 & 1 \end{pmatrix}$$

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Fundamental domain
$\psi(x, y, z) \mapsto (\{x\}, \{y\}, \{z - x\lfloor y \rfloor\})$. Can take Lipschitz function $F(x, y, z) = e(\psi_3)\varphi(\{y\})$ for cutoff φ.
$F(g(n)\Gamma) = e^{2\pi i(-\alpha n\lfloor \beta n\rfloor + \binom{n}{2}\alpha\beta + \gamma n)}\varphi(\beta n)$.

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- Even simpler $n \mapsto e^{2\pi i \alpha n}$.

Applying the Generalized von Neumann Theorem

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or (L. 2023)

$$\|\Lambda - \Lambda_Q\|_{U^3([N])} \ll_A^{\text{ineff}} \log^{-A}(N).$$

A couple of obstructions remain.

Obstructions to quantitative bounds

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- Siegel's theorem is ineffective (subtle issue).
  - Can fix this by subtracting the "contribution" of the possible Siegel zero $\chi_{Siegel}(n)n^{\beta-1}\Lambda_Q$ (where β is the Siegel zero).
  - Can evaluate $\|\chi(n)n^{\beta-1}\Lambda_Q\|_{U^{s+1}([N])} \sim \|\chi\|_{U^{s+1}([N])}$ directly to obtain cancellation.

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  - Can evaluate $\|\chi(n)n^{\beta-1}\Lambda_Q\|_{U^{s+1}([N])} \sim \|\chi\|_{U^{s+1}([N])}$ directly to obtain cancellation.
  - For simplicity, assume Landau-Siegel zeros don't exist.

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- In (L. 2023), show $\|\Lambda - \Lambda_Q\|_{U^3([N])} \ll \exp(-O(\log(N)^c))$.

Applying the Inverse Theorem

It can be reduced to show (Tao-Teräväinen 2021):

$$\mathbb{E}_{n\in[N]}(\Lambda - \Lambda_Q)(n)F(g(n)\Gamma) \ll \exp(-O(\log^c(N)))$$

for $g(n)\Gamma$ having dimension $d = \log\log(N)^c$ and complexity $\exp(O(\exp(d^{O(1)})))$

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where $d = \log(N)^c$ and complexity $\exp(O(d)^{O(1)})$ (actually can take complexity to be $O(d)^{O(1)}$).

Type I and type II reduction

Using Vaughan's decomposition, we are reduced to showing for *certain* nilsequences

- Type I estimate:

$$\mathbb{E}_{n\in[N], n\equiv 0 \pmod d} F(g(n)\Gamma) \ll \exp(-O(\log^c(N)))$$

  for "most" $d \in [D, 2D]$ with $D \leq N^{2/3}$ (actually, for (L. 2023) must take $D \leq \exp(O(\log^c(N)))$)

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- Type II estimate: for A, D with $AD \sim N$ and $N^{1/3} \leq D \leq N^{2/3}$

$$\mathbb{E}_{a,a' \in [A,2A], d, d' \in [D,2D]} F(g(ad)\Gamma)\overline{F(g(a'd)\Gamma)F(g(ad')\Gamma)}$$

$$F(g(a'd')\Gamma) \ll \exp(-O(\log(N)^c)).$$

Vinogradov's Proof

Vinogradov's Proof boils down to showing that if α is "very irrational", then

$$\mathbb{E}_{n \in [N], n \equiv 0 \pmod{d}} e^{2\pi i \alpha n} \ll \exp(-O(\log^c(N)))$$

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for "most" $d \in [D, 2D]$ or

$$\mathbb{E}_{a, a' \in [A, 2A], d, d' \in [D, 2D]} e^{2\pi i \alpha (a - a')(d - d')} \ll \exp(-O(\log(N)^c)).$$

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Vinogradov's Proof

Vinogradov's Proof boils down to showing that if α is "very irrational", then

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This shows that α can't be "very irrational." Similar computation for type II.

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If $\alpha = O(1/N) \pmod 1$, expect sum to be large. Otherwise, expect sum to be small.

Example 2

Let $F : S^1 \to \mathbb{C}$ be smooth

$$\mathbb{E}_{n \in [N]} F(\alpha n) = \int_{S^1} F(\theta) d\theta + [\text{Error}]$$

Expect error term to be small if α is irrational.

Example 3

Let $F : \mathbb{T}^d \to \mathbb{C}$ be smooth.

$$\mathbb{E}_{n \in [N]} F(\alpha_1 n, \alpha_2 n, \ldots, \alpha_d n) =$$

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For generic $(\alpha_1, \dots, \alpha_d)$ expect error to be small. It's possible that $(\alpha_1, \dots, \alpha_d)$ can lie in a **subgroup**. This can make the error large.

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- Green-Tao give a quantitative equidistribution theorem.
- Tao-Teräväinen work out explicit bounds for Green-Tao, obtaining losses double exponential in dimension.

Improvements over Tao-Teräväinen

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- Tao-Teräväinen also lose two logarithms from the equidistribution theory of nilmanifolds.
- Inserting Sanders' result makes those two logarithms loss into a one logarithm loss.
- Gowers-Wolf (2010) (and also Green-Tao (2007, 2017)) give a way to equidistribute on two-step nilmanifolds without that one logarithm loss.

More details about the proof

- Gowers-Wolf's approach tells you that for "many" $d \in [D, 2D]$, $F(g(ad)\Gamma)$ is roughly constant on some Bohr set.

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where $B_d = \{n \in B : n \equiv 0 \pmod d\}$ and $\mathcal{D}$ are the values of d that $F(g(ad)\Gamma)$ are constant and B the Bohr set.
- Seems difficult to show. (In $\mathbb{F}_p[T]$, Bienvenu and Le need bilinear Bogolyubov and some tricky matrix analysis)

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- In a type II sum, can get “cancellation” in both a and d variables, allowing one to prove something stronger: that $F(g(ad)\Gamma)$ is roughly a “rational phase with bounded denominator” along B_d for **every** d in some interval $I = [K, 2K]$.

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- Can estimate $\bigcup_{d \in I} B_d$ efficiently via the second moment method by restricting to the primes in I.
- For the type I sum, convert the case of when d is large to the type II case.
- For d small in the type I case, get enough cancellation in one variable to prove the theorem anyways.

Remarks and Loose Threads

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  Unclear in that framework what happens when you sample through many d.
- Can this method generalize to higher step nilsequences to obtain single exponential losses in dimension?

Update

- L. has proved a generalized equidistribution of nilsequences that gives good bounds for higher degree nilsequences. See https://arxiv.org/abs/2306.13820.
- Conditional on the quasi-polynomial $U^{s+1}(\mathbb{Z}/N\mathbb{Z})$ inverse theorem, the higher order Möbius and von Mangoldt uniformity estimates with similar bounds can be shown.

Thank you!

Joni Teräväinen's slides: https://drive.google.com/file/d/1DdvWyGVlCjvpM0yEr-q0lqs4o0fcBKuw/view